Localized vortices with a semi-integer charge in nonlinear dynamical lattices

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The topological charge S of vortexlike configurations in two-dimensional (2D) dynamical lattices need not necessarily be integer, nor is it a dynamical invariant. Accordingly, we demonstrate that the discrete nonlinear Schrödinger (DNLS) equation in 2D has stationary solutions in the form of a vortex with $S=1/2$, which does not exist in the model's continuum counterpart. Analysis of the DNLS equation linearized about the vortex shows that it is stable except for, possibly, *extremely weak* instabilities (at the level of numerical precision). Direct simulations of the full DNLS model in 2D show that the $S=1/2$ vortex soliton is a stable oscillating solution. This behavior of classical dynamical lattices is in contrast with a recently reported result by Clay *et al.* [Phys. Rev. Lett. **86**, 4085 (2001)], according to which fractional charges in *quantum* lattices are subject to dynamical rearrangement into integer charges. We also consider $S=1$ discrete vortices that may be built as a pair of $S=1/2$ ones. These are different from the cross-shaped $S=1$ vortices that were recently found in the same 2D model. The $S=1$ vortices found in this work have larger energy and a slightly smaller stability range. We also find an analog of the $S=1/2$ vortices in the 1D DNLS model, which also turns out to be a stable oscillating soliton, different from the twisted localized modes recently found in the 1D model.

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I. INTRODUCTION

Nonlinear lattice equations naturally arise as models of various physical systems, and are also a subject of interest in their own right as an important class of nonlinear dynamical systems [1,2]. A fundamental lattice model is represented by the discrete nonlinear Schrödinger (DNLS) equation with cubic on-site nonlinearity, which finds straightforward applications, both theoretical [3] and experimental [4], in the transverse dynamics of arrays of optical waveguides (fibers) with the Kerr nonlinearity. It is also used as a generic asymptotic envelope equation for nonlinear discrete equations of the Klein-Gordon (including sine-Gordon) type [5], which model physical phenomena ranging from Josephson junctions [6] to the local denaturation of the DNA double strand [7].

The one-dimensional (1D) DNLS equation is

$$i\dot{\psi}_n=-C\Delta_2\psi_n-|\psi_n|^2\psi_n, \tag{1}$$

where ψ_n is the dynamical field, n is the lattice discrete coordinate, and $\Delta_2\psi_n\equiv\psi_{n+1}+\psi_{n-1}-2\psi_n$ is the 1D discrete Laplacian. The coupling constant C is related to the lattice spacing h by

$$C=1/h^2. \tag{2}$$

The 2D version of the model is

$$i\dot{\psi}_{m,n}=-C\Delta_2\psi_{m,n}-|\psi_{m,n}|^2\psi_{m,n}, \tag{3}$$

where the 2D discrete Laplacian is $\Delta_2\psi_{m,n}\equiv\psi_{m+1,n}+\psi_{m,n+1}+\psi_{m-1,n}+\psi_{m,n-1}-4\psi_{m,n}$. Both the 1D and 2D equations can be derived from a Hamiltonian. In the 2D case, it is

$$H=\sum_{m,n}\left[C(|\psi_{m+1,n}-\psi_{m,n}|^2+|\psi_{m,n+1}-\psi_{m,n}|^2)-\frac{1}{2}|\psi_{m,n}|^4\right]. \tag{4}$$

It is customary to seek standing-wave solutions to Eqs. (1)–(3) in the form

$$\psi_{m,n}=\exp(i\Lambda t)u_{m,n}, \tag{5}$$

where Λ is the frequency of the solution. Recently, localized 1D solutions subject to the symmetry constraint $u(-n)=-u(n)$, with $n=1,2,3,\ldots$ (where the lattice site with $n=0$ either has $u_0\equiv0$, or is dropped by definition) were found [8]. Their linear stability and dynamical properties were analyzed in Ref. [9]. These *twisted localized modes* (TLMs) were then used to construct stable two-dimensional discrete vortex solitons in Ref. [10] (2D vortices were mentioned in some earlier works [2,11], but their detailed structure and stability were not studied before). In particular, the fact that the TLM modes have an inherent π phase difference between the regions $n\rightarrow-\infty$ and $n\rightarrow+\infty$ was crucial in constructing the vortices in Ref. [10]; namely, by using a set of two such modes, one in the real part of the solution along one of the two spatial directions, and another one in the imaginary part along the orthogonal direction, a 2π phase difference may be lent to the entire pattern. This procedure makes it possible to generate a discrete cross-shaped vortex soliton, whose structure qualitatively resembles the continuum-limit vortex wave form $\sim\exp(i\theta)$, θ being the polar angle in the lattice plane. The investigation of the linear stability of these solitons in Ref. [10] revealed that the TLM structure of their constituents may give rise to an oscillatory instability at large values of the coupling constant C (this instability is similar to those considered in Refs. [9,12]). However, the cross-shaped vortices are *stable* if the coupling constant is smaller than a certain critical value, $C<C_{\mathrm{cr}}$, i.e., if the model is not too close to its continuum limit. Vortices with the double topological charge were also found in Ref. [10], but they all turned out to be unstable; it is very plausible that all the higher-order vortices are unstable too. It should be noted, in passing, that it has been conjectured (work toward proving that conjecture is currently in progress [13]) that the presence of the π phase shift in TLMs and, in general, in pulse-antipulse bound states, of which TLMs are a special case, is generically related to potential oscillatory instabilities.

Thus, recent work [8–10] has revealed the possibility of existence of stable solutions with an integer (in fact, unitary) topological charge in 2D dynamical lattices. A nontrivial aspect of this result is that, contrary to the continuum models, in lattice systems the topological charge is *not* a dynamical invariant. Therefore, the very existence of such stationary solutions is not obvious. Moreover, in continuum models localized vortices can easily be found, but in most cases they are strongly unstable. Only nonlinearities of a special type, such as those combining quadratic and self-defocusing cubic terms, make it possible to construct families of localized vortices that are stable in a broad parametric region [14].

Since the definition of the topological charge in dynamical lattices is less precise than in continuum media, the question naturally arises as to whether *fractionally charged* vortices are possible in 2D lattices. In fact, we will focus solely on the single noninteger value of the topological charge (or vorticity) $S=1/2$. Upon finding such solutions, we will investigate their stability by using them as initial conditions in direct numerical simulations.

An additional motivation for this work is the recently published results concerning fractionally charged states in *quantum* lattice models [15]. That work aimed to explain the absence of experimental observation of fractionally charged solitons in π-conjugated polymers and charge-density-wave solids. It was found that, once fractional charges were introduced "by hand" in the quantum-lattice models considered, further dimerization of the semi-integer charges into integer ones inevitably followed, as the system evolved in time. We will see herein that the same result is not true for classical Hamiltonian dynamical lattices, such as that governed by the DNLS equation. Instead, we find *stable* oscillating 2D solitons with semi-integer vorticity. This result prompts us then to reconsider the 1D case; as a result, we also find stable 1D counterparts of the $S=1/2$ vortex solitons. Additionally, in the 2D lattice we study integer-charged vortices (with $S=1$) to which the semi-integer-charged ones would recombine if they were subject to reintegerization. These prove to be stable localized solutions different from the cross-shaped ones considered in Refs. [9,10].

The paper is organized as follows. In Sec. II we present stationary solutions in the form of localized vortices in two dimensions with the semi-integer topological charge $S=1/2$, and also the stable vortices with $S=1$ into which the $S=1/2$ ones would recombine if they were unstable. These $S=1$ stable vortices exist despite the fact that the recombination does not actually take place. In Sec. III, following the analogy with the 2D problem, similar semi-integer-charged solutions are studied in a 1D DNLS model, and are also found to be stable oscillatory solutions. The results are summarized in Sec. IV.

II. SEMI-INFINITE-CHARGE VORTICES, AND $S=1$ VORTICES, IN THE TWO-DIMENSIONAL LATTICE

A. Revisiting discrete localized vortices with an integer charge

Before considering the fractionally charged lattice vortices in detail, it is relevant to summarize the approach to the study of integer-charge ones, developed in Ref. [10]. In that work a dual twisted ansatz was considered, consisting of a TLM in the real and a TLM in the imaginary parts of the solution, a configuration with a 2π phase shift nested in it, hence bearing a nontrivial vorticity. This configuration was fed, as an initial guess, into a Newton solver of the system of nonlinear algebraic equations

$$\Lambda u_{m,n}=C\Delta_2 u_{m,n}+|u_{m,n}|^2 u_{m,n}, \tag{6}$$

which results from the substitution of the ansatz (5) into Eq. (3). Notice that one of the parameters Λ and C can be scaled out from Eq. (6); however, for the sake of clarity of the results to be obtained below, we keep both parameters, fixing Λ and studying the behavior of solutions as a function of C.

The Newton iterations typically converge up to a prescribed accuracy [usually $O(10^{-8})$]. Once the solution was obtained, linear stability analysis was performed, substituting $\psi_{m,n}=\exp(i\Lambda t)[\psi_{m,n}+v_{m,n}(t)]$ with infinitesimal perturbations $v_{m,n}(t)$ into Eq. (3), to derive the linearized equation for $v_{m,n}$,

$$i\dot{v}_{m,n}+C\Delta_2 v_{m,n}+2|u_{m,n}|^2 v_{m,n}+u_{m,n}^2 v_{m,n}^*-\Lambda v_{m,n}=0, \tag{7}$$

where * denotes complex conjugation. Looking for solutions of Eqs. (7) in the form $v_{m,n}=a_{m,n}\exp(-i\omega t)+b_{m,n}\exp(i\omega^* t)$ leads to an eigenvalue problem for ω [10], which can be solved numerically. It was thus found that the discrete vortex solitons with the unitary topological charge

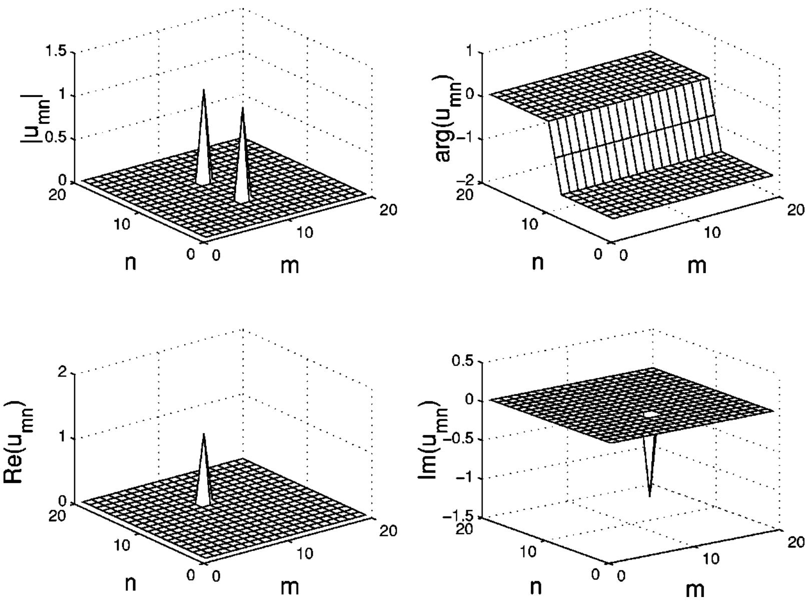

FIG. 1. The initial configuration generating a fractionally charged solution with $S=1/2$ for $\Lambda=0.32$ and $C=0.001$. The top left and right panels show the absolute value and phase of the field, respectively. The bottom left and right panels show the real and imaginary parts of the field, respectively.

$S=1$ are stable for $C<C_{\mathrm{cr}}$. For instance, $C_{\mathrm{cr}}(\Lambda=0.32)\approx 0.126$. If $C>C_{\mathrm{cr}}$, one of the eigenvalues ω collides with the continuous spectrum and, due to its opposite Krein signature (see, e.g., Refs. [2,9,12,16,17]), the corresponding bifurcation gives rise to a quartet of eigenvalues which generate an oscillatory instability. This is the so-called Hamiltonian Hopf bifurcation [18]. It should be remarked that this instability scenario is inherited from the 1D constituents of the 2D discrete vortex (i.e., two orthogonally oriented TLMs). In fact, the 2D nature of the vortex does not strongly alter the coupling strength at which the instability occurs, in comparison with the 1D counterpart of the TLM type. In particular, the 2D critical value is to be compared to the critical value $C_{\mathrm{cr}}=0.138$ found for the same frequency $\Lambda=0.32$ in the 1D system.

B. Vortices with the semi-integer topological charge

A vortex with a topological charge $S=1/2$ can be formally sought for in a continuum medium as a solution $\sim\exp(i\theta/2)$. In fact, in the continuum setting it cannot exist because physical fields (not only the phase) are necessarily discontinuous in such a formal ansatz, and hence the gradient part of the corresponding Hamiltonian diverges, as it contains a nonintegrable term proportional to a squared δ function. On the contrary, in the discrete counterpart of the continuum medium, the Hamiltonian remains finite, but it will take very large values for the same reason, leaving very little chance for stability of such a high-energy configuration (and also for its existence as a stationary state).

A more promising continuum-model ansatz that may help to identify $S=1/2$ vortex solitons in the lattice is

$$u\sim\exp(\pm i|\theta|/2), \tag{8}$$

where the angular variable θ is defined so that it takes values $-\pi<\theta<\pi$. A lattice solution of this type can be created, adopting the initial-guess ansatz

$$\mathrm{Re}(u_{10,n})=(\,\ldots,0,0,1,0,\ldots\,),$$

$$\mathrm{Im}(u_{10,n})=(\,\ldots,-1,0,0,0,\ldots\,) \tag{9}$$

(. . . stands for zeros) along (say) the row $m=10$ in the 2D lattice, the field being set equal to zero everywhere else. It is obvious that this dipolelike configuration (shown in Fig. 1) may indeed be regarded as one conforming to the expression (8) with the lower sign, provided that the center of the configuration is set at the point $(m,n)=(10,10)$ between the two sites at which the real and imaginary parts of the initial configuration (9) are concentrated.

Solutions generated by initial configurations of the type shown in Fig. 1 were constructed both by the direct Newton method applied to Eqs. (6), and by continuation from the anticontinuum limit of $C=0$. The Newton method showed that, for large values of the coupling constant C, the solution always tended to become symmetric, with both its real and imaginary parts taking the form of a TLM, despite the asymmetry of the initial configuration. To ensure that the iterations nevertheless converge to the asymmetric solution sought for (if it exists), the following "enforced-convergence" method was used: the imaginary and real parts of the field at the sites where, respectively, the real and imaginary parts of the field were originally set equal to ± 1, were set equal to 0 ("by hand") in each iteration. In this way, we effectively pushed every Newton iteration closer to the basin of attraction of the $S=1/2$ vortex.

The resulting continuation diagram is shown in Fig. 2. The bottom panel shows that the norm of the solution increases as a function of the coupling constant C, which can be understood as a result of involving more lattice sites in the localized vortex state with the increase of C. Despite the use of the above-mentioned "enforced-convergence" method, the solution branch could not be continued to values of C exceeding some critical value C_{cr}. For instance, $C_{\mathrm{cr}}=2.25\times 10^{-3}$ for $\Lambda=0.32$, which is the largest value of C in Fig. 2. Beyond this critical point, the iterations always gave

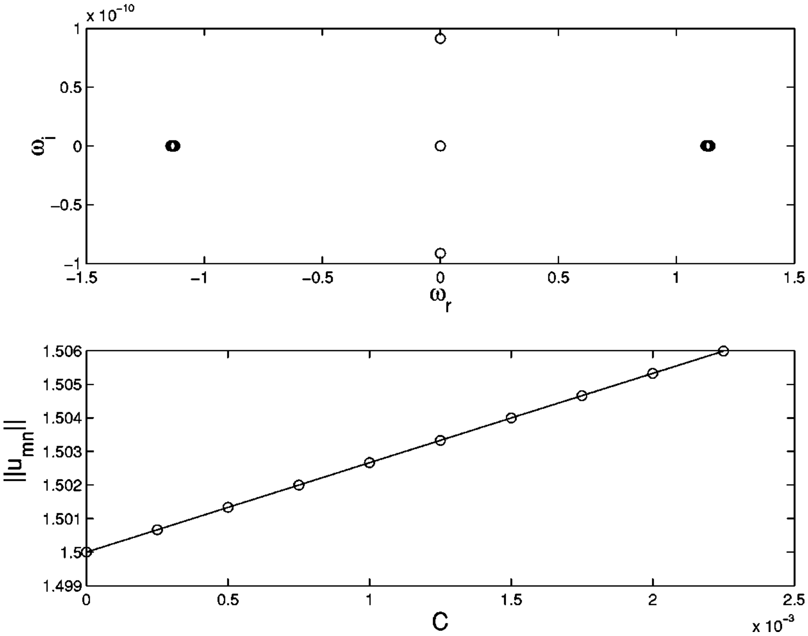

FIG. 2. The bottom panel shows the result of continuation of the $S=1/2$ localized vortex generated by the initial configuration displayed in Fig. 1, in terms of the solution's norm $\sqrt{\Sigma_{m,n}|u_{m,n}|^2}$, vs the coupling constant C. It was not possible to continue the branch of the $S=1/2$ (*asymmetric*, see Fig. 2) solution beyond the upper-bound value $C=2.25\times10^{-3}$ marked in the bottom panel, despite using the "enforced-convergence" technique detailed in the text. For $C>2.25\times10^{-3}$, the Newton method always converges to a different, *symmetric* solution of the TLM type (which was mentioned in the text), instead of the asymmetric vortex. The eigenvalues produced by the linear stability analysis of the solution corresponding to the last point in the bottom panel (i.e., for $C=2.25\times10^{-3}$) are shown in the top panel, where ω_r and ω_i stand for the real and imaginary parts of the eigenfrequency ω of the linear stability problem.

rise to a symmetric stationary configuration containing TLMs in the real and imaginary parts of the solution. The topological charge of this configuration can be identified as $S=1$; however, both its constituent TLMs are oriented along one direction, in contrast with the crosslike $S=1$ vortex that was studied in Ref. [10] and displayed in the previous section. We will return to this different type of $S=1$ vortex below. Thus, the $S=1/2$ vortices may exist only in strongly discrete lattices, with the spacing $h>C^{-1/2}\approx 21$ (for $\Lambda=0.32$), according to Eq. (2).

The linear stability analysis of the vortices with the half-integer topological charge always produced a formally unstable pair of eigenvalues. However, these are found at the limit of the numerical precision; see the top panel in Fig. 2.

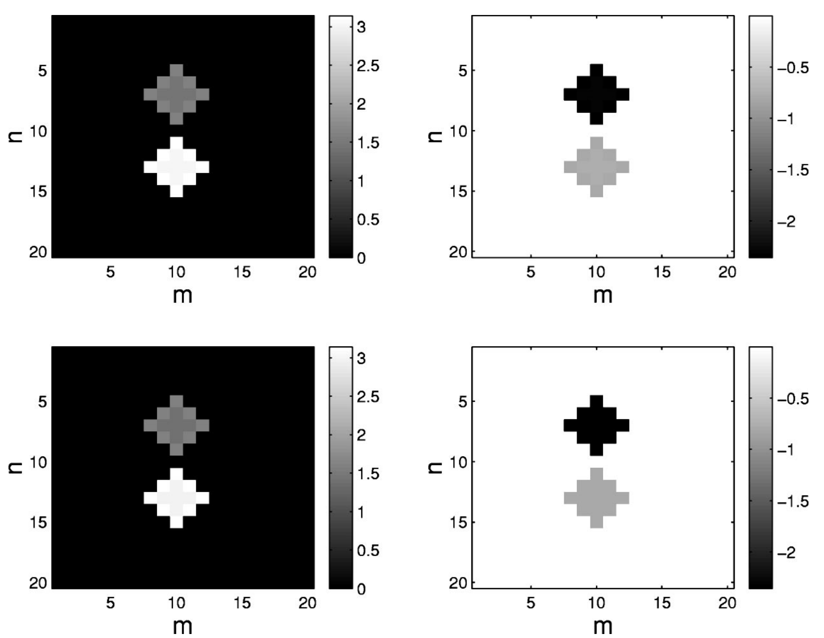

FIG. 3. The evolution of the $S=1/2$ vortex is shown, for $C=0.001$ and $\Lambda=0.32$, in terms of the contour plots of the phase field, starting from the numerically exact stationary vortex. The top left, top right, bottom left, and bottom right panels display the phase fields at the moments of time $t=225$, $t=540$, $t=850$, and $t=1170$, respectively. A phase change of $\approx\pi$ is observed in a contour surrounding the solution in all four panels, verifying the persistence of the $S=1/2$ vortex. In the top left panel, the gray scale shows the variation of the phase between ≈ 0 (the lightest color of the pattern) and $\approx\pi$ (the darkest gray color of the pattern); similarly for the bottom left panel. For the right panels, the phase has a difference between ≈ 0 (the light gray color of the pattern) and $\approx -\pi$ (the black color in the pattern), which is again a phase difference of π. The reversal of the gray scale is a mere artifact of the plotting program; what is really important, in all cases, is that a phase change of $\approx\pi$ can be observed.

This means that even if this instability is not spurious (which cannot be definitely claimed within the available numerical accuracy), it will become manifest only at extremely large times ($\gtrsim 10^{10}$), which are not relevant for the computational time scales probed here, and may also be irrelevant for physical applications.

To observe dynamical evolution of the fractionally charged vortices, we simulated the full nonlinear equations (3), starting with the initial configuration in the form of the ansatz (9). The results are displayed in Fig. 3 in terms of the phase of the complex solution. It is clearly seen that the net phase change of π present in the initial condition is preserved during the simulation time. Detailed consideration of the solution (data not shown) demonstrates that the phase differences of $\pi/2$ between the top and bottom sides of the pattern are maintained in both the real and imaginary parts of the solution. Hence, the total phase shift nested in the fractionally charged discrete vortex is π, as can be directly observed in the four panels of Fig. 3. A similar stable oscillatory behavior is observed in the 1D analog of the solution; see Sec. III below.

It is relevant to stress again that the topological charge of the vortex may change in time, as it is not a dynamical invariant in the lattice system. The actual conservation of the initial semi-infinite charge in the 2D DNLS model, demonstrated above, can be contrasted with the reintegerization of fractional charges in quantum lattices, reported in Ref. [15]. Thus we conclude that the $S=1/2$ vortices in the 2D DNLS model maintain their fractional-charge nature for very long times, suggesting the possibility of experimental observation of these vortices, albeit in strongly discrete (weakly coupled) lattice systems.

As mentioned above, for $C>2.25\times 10^{-3}$ (and $\Lambda=0.32$) the Newton iterations always converge not to the $S=1/2$ vortices, but rather to $S=1$ ones, which are, however, different from the cross-shaped vortices found in Ref. [10]. As was also mentioned above, these vortices consist of two aligned TLMs in the real and imaginary parts of the solution, which produce a total phase change of 2π on a contour around the solution. The fact that such solutions were not considered in previous work [10] prompted us to study them in more detail. In particular, it was found that these vortices are *less* energetically favorable than the "more two-dimensional" crosslike $S=1$ stable vortices found in Ref. [10], in which the real and imaginary parts of the solution consist of two quasi-1D TLMs oriented along two orthogonal lattice directions. For example, for $\Lambda=0.32$ and $C=0.1$, the $S=1$ vortex found here (see the lower-row panels in Fig. 3) has the energy [calculated as the value of the Hamiltonian (4)] $E_{\mathrm{new}}=-0.899$, while the crosslike $S=1$ vortex from Ref. [10] gives rise to the energy value $E_{\mathrm{old}}=-1.794$ for the same values of Λ and S. In accord with this, the range of stability of the $S=1$ vortices found here is smaller (although not much smaller) than that of the previously studied cross-shaped ones. In particular, we have found that, for the fixed frequency $\Lambda=0.32$, an oscillatory instability of the present vortex sets in at $C\approx 0.11$, which is to be compared to a similar instability threshold $C\approx 0.126$ for the crosslike vortex found at the same value of the frequency Λ.

Finally, we note that the existence of the two distinct types of discrete $S=1$ vortex, with the angle 0 (here) and $\pi/2$ (in the previously known solution) between the constituent quasi-1D TLM components in their real and imaginary parts, also prompted us to look for discrete vortices composed of two quasi-1D TLMs oriented at different angles, in particular by placing one TLM along a lattice axis and the other TLM along a diagonal. However, at least for typical values of the coupling constant C for which the vortices considered above exist, the Newton iterations did not converge to stationary configurations with such an initial shape, even though they can be found very close to the anticontinuum limit (at extremely small values of C); this is, of course, to be expected, as in the limit $C=0$ any configuration is a formal solution.

III. ANALOGS OF $S=1/2$ VORTICES IN THE ONE-DIMENSIONAL LATTICE

Given the findings of the previous section, it is natural to reconsider the 1D model, and search for 1D analogs of the $S=1/2$ vortices found in the 2D lattice; these should be different from the previously considered TLMs [8,9]. To construct such 1D modes, we use techniques similar to those applied to the 2D model. In the 1D lattice (containing 100 sites), we start from the anticontinuum limit ($C=0$), with the imaginary part of the field having values -1 at, say, the site $n_i=47$ and zero elsewhere, and the real part having values 1 at, say, $n_r=53$ and zero elsewhere. As would be expected, the larger initial distance between the constituent pulses gives the possibility of a wider (in terms of C) domain of existence of the 1D patterns sought for.

The norm of the stationary solution thus found is shown in the left panel of Fig. 4 as a function of C. The enforced-convergence method described in the previous section was also used here in the following form: the imaginary part of the field at $n_r=53$, and the real part at $n_i=47$, were set to zero "by hand" in every iteration of the Newton method. By means of this approach, a branch of the 1D analog of the $S=1/2$ vortex solution could be continued up to $C\approx 0.025$. However, as can be seen in the right panel of Fig. 4, which shows the imaginary part of the field at the sites $n=51$ and 52, which should be almost zero for the solution considered, the imaginary part of the field at the latter site actually already starts to diverge at $C\approx 0.01$. From this value of C onward, the basin of attraction of the 1D analog of the $S=1/2$ vortex becomes very narrow.

The time evolution of this 1D solution is shown in Fig. 5, which shows the evolution of the real and imaginary parts of the field at the sites $n_i=47$ and $n_r=53$. Similar to what was observed for the 2D vortices with $S=1/2$, their 1D analogs are also dynamically stable: as seen in Fig. 5, the $S=1/2$ solution in the 1D lattice is characterized by harmonic oscillations of its intrinsic phase, while the local powers (squared amplitudes) of the field, $|u_n|^2$, are time independent.

IV. CONCLUSIONS

The aim of this work was to search for *localized* vortex-like solutions in two-dimensional nonlinear dynamical lat-

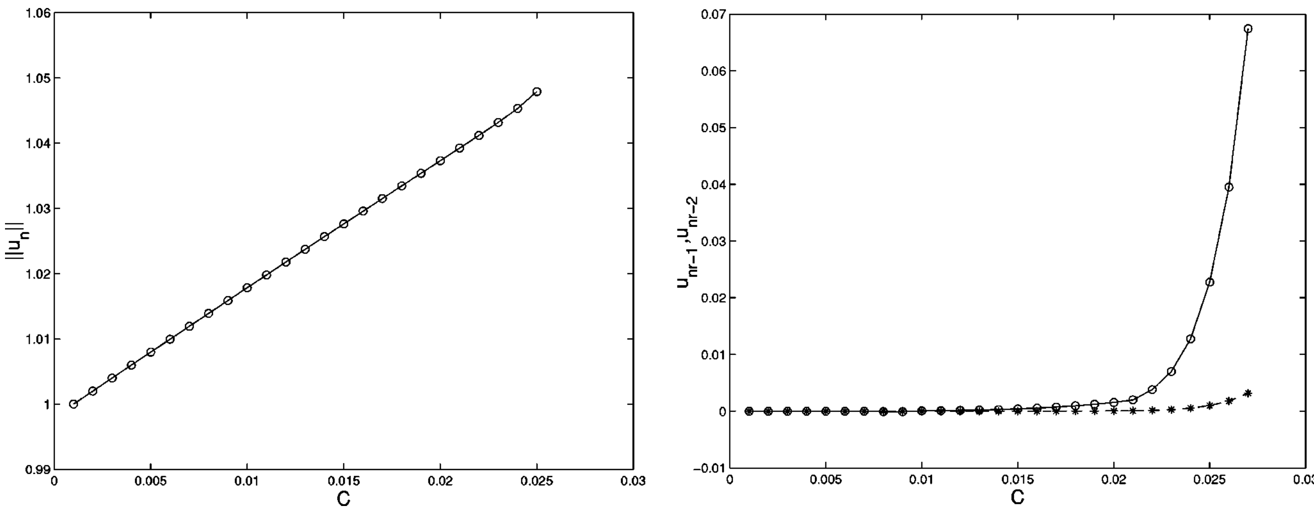

FIG. 4. In the left panel, the norm $\sqrt{\Sigma_n |u_n|^2}$ of the 1D counterpart of the $S=1/2$ vortex is shown as a function of the coupling constant C. The right panel shows the C dependence of the imaginary part of the field at the sites n_r-1 (circles stand for the data points and are connected by a solid line) and n_r-2 (stars are the data points and are connected by a dashed line). It is seen from the right panel that, at $C>0.01$, the imaginary part of the field at the site n_r-1 starts to diverge, and the system tends to become more symmetric, turning into a pattern consisting of TLMs in both the real and imaginary parts of the solution.

tices with a fractional (in fact, semi-integer) topological charge (“discrete vorticity”) S, which may, generally, be possible in discrete media, in contrast with continua. By means of numerical methods, we have demonstrated that such a stationary solution, with the topological charge $S=1/2$, indeed exists in the 2D discrete cubic nonlinear Schrödinger equation. Analysis of the equation linearized about the $S=1/2$ vortex shows that, even though this structure may be subject to an instability, such an instability would be extremely weak and, in any case, not relevant for the time scales considered here. The instability, if any, would not be relevant for any physical application either, which makes the $S=1/2$ localized vortices physically meaningful objects. Direct simulations of the full equation have shown that these vortices are, indeed, stable, preserving their semi-integer vorticity. This result can be contrasted to a recently reported

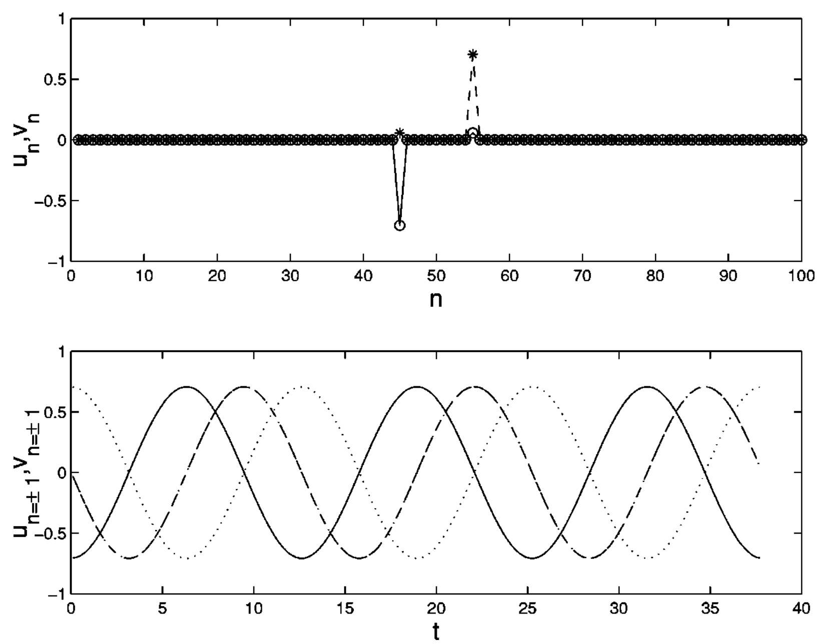

FIG. 5. Time evolution of the real and imaginary parts of the field in the 1D $S=1/2$ solution at the sites n_r and n_i is displayed in the bottom panel. The initial configuration was the exact asymmetric solution (from Fig. 4) for $C=0.005$. The dotted, solid, and (overlapping) dashed and dash-dotted curves show, respectively, the time evolution of the real part of the field at the site n_r, the imaginary part at n_i, the imaginary part at n_r, and the real part at n_i. Notice the $\pi/2$ phase difference between the oscillations of the real and imaginary parts of the field at both sites n_r and n_i. The spatial profile of the field corresponding to the last instant of the time evolution is shown in the top panel. The solid line with circles and the dashed one with stars show, respectively, the imaginary and real parts of the solution.

finding, according to which initially seeded fractional charges in quantum lattice models dynamically rearrange themselves into integer charges. Beyond a critical value of the coupling constant, we were unable to continue the $S=1/2$ branch; the Newton iterations converge in this case to a different type of integer-charged $(S=1)$ vortex soliton, consisting of two aligned quasi-1D twisted localized modes, carrying the real and imaginary parts of the solution. The resulting $S=1$ discrete vortices are quite different from the cross-shaped ones, that were recently found in the same 2D model, as the $S=1$ vortices found in the present work have larger energy and a smaller stability range. They may also be stable, nevertheless. We have also found a counterpart of the $S=1/2$ vortex in the one-dimensional discrete nonlinear Schrödinger equation. The latter pattern is a stable localized state with intrinsic phase oscillations, which is different from the 1D twisted localized modes that are known in the lattice model.

An interesting topic for future investigations would concern the identification of the basin of attraction of solutions with fractional charge. In particular, it would be interesting to identify whether instabilities of solutions with a different topological charge (or even without topological charge, such as, for instance, multiple pulses without phase difference) could give rise to solutions with $S=1/2$. Naturally, from the considerations presented above, one can infer that such a basin of attraction would be larger for smaller C. A detailed investigation of these and related questions is currently under way.

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